# MATH 207(c)–Analysis IV



**1.** Find the coefficients of the Taylor series of the following functions around the given point:

**(a)** $f(z) = e^z$ at $z_0 = 0$.

**(b)** $f(z) = e^z$ at $z_0 = \pi i$.

**(c)** $f(z) = e^{z^2}$ at $z_0 = 0$.

**(d)** $f(z) = \cos(z-3)$ at $z_0 = 3$.

**(e)** $f(z) = \big(\sin(z)\big)^2$ at $z_0 = 0$.

**2.** For the functions below, compute the Laurent series at the specified point, determine the radius of convergence and specify the nature of the singularity (i.e. regular point, pole or essential singularity):

**(a)** $g(z) = \frac{e^z}{(z-2)^2}$ at $z_0 = 2$.

**(b)** $g(z) = \frac{1}{z(z+1)}$ at $z_0 = -1$.

**(c)** $g(z) = \frac{z^3-z+2}{(z-i)^2}$ at $z_0 = i$.

**(d)** $g(z) = \frac{\cos\big((z-1)^2\big)}{(z-1)^3}$ at $z_0 = 1$.

**3.** For the functions below, compute the Laurent series at the specified point, determine the radius of convergence and specify the nature of the singularity (i.e. regular point, pole or essential singularity):

**(a)** $f(z) = z\cos(\frac{1}{z})$ at $z_0 = 0$.

**(b)** $f(z) = \frac{\sin(z)}{(z-\pi)}$ at $z_0 = \pi$.

**(c)** $f(z) = \frac{z^{\frac{1}{2}}}{(z-1)^2}$ at $z_0 = 1$.

**4.** Let $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$ be the standard parametrization of the unit circle centered at the origin. Compute the following integrals:

**(a)** $A = \int_\gamma \frac{e^z}{z^2(z-2)}\,dz$.

**(b)** $B = \int_\gamma \frac{\sin(z)}{z(z-2i)}\,dz$.

**(c)** $C = \int_\gamma \frac{z^3-2z}{z(z-10)}\,dz$.

**(d)** $D = \int_\gamma \left(\frac{1}{z^2} + \frac{1}{z} - e^z \sin(z)\right) dz.$

**5.** Consider the function $f(z) = \log(1 + z^2)$. What is the maximal subset of $\mathbb{C}$ on which $f$ is holomorhic? Compute the Taylor series at $z_0 = 0$. What is it's radius of convergence?

**6.** In this exercise, we will prove Liouville's theorem, which states that if $f : \mathbb{C} \to \mathbb{C}$ is an entire function which is bounded (i.e. there exists some $M > 0$ such that $|f(z)| \leqslant M$ for all $z \in \mathbb{C}$), then $f$ must be constant.

(a) Show that, for any $z_0 \in \mathbb{C}$, if $\gamma_R(t) = z_0 + Re^{it}$ for $t \in [0, 2\pi]$, then

$$|f'(z_0)| \leqslant \int_0^{2\pi} \frac{\big|f(\gamma_R(t))\big|}{R}\, dt.$$

*(Hint: Use Cauchy's integral formula.)*

(b) Assuming that $f$ is bounded, show that $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$. Hence, $f$ is constant.

**7.** Let $\mathcal{D} \subseteq \mathbb{C}$ be open and simply connected. Let $f : \mathcal{D} \to \mathbb{C}$ be holomorphic. For any $z_0 \in \mathcal{D}$ and any $R > 0$ such that the closed disk

$$\overline{B_R(z_0)} = \Big\{ z \in \mathbb{C} : |z - z_0| \leqslant R \Big\}$$

is contained inside $\mathcal{D}$, show that the following identity holds:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\big(z_0 + Re^{it}\big)\, dt.$$

The above is known as the *mean value property* for holomorphic functions.
*(Hint: Start from Cauchy's integral formula on a suitable curve, and calculate the integral explicitly.)*